**Problem 1.** Find two linearly independent solutions of the equation

$$2ty'' + y' + ty = 0 \qquad 0 < t < \infty \tag{1}$$

*Solution.* We notice that $t = 0$ is a singular point, therefore we can apply Froebenius Method. This method consists on looking for a power series solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}, \qquad a_0 \neq 0 \tag{2}$$

We differentiate (2) twice to get

$$y'(t) = \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} \tag{3}$$

and

$$y''(t) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2}. \tag{4}$$

We replace these equations back into (1):

$$t^r \left[ 2\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n-1} + \sum_{n=0}^{\infty} (n+r) a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^{n+1} \right] = 0 \tag{5}$$

We rearrange the last sum of the left hand side of (5) to get

$$t^r \left[ 2\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n-1} + \sum_{n=0}^{\infty} (n+r) a_n t^{n-1} + \sum_{n=2}^{\infty} a_{n-2} t^{n-1} \right] = 0 \tag{6}$$

Now, we almost can sum all together the three sums, but the third one starts at $n = 2$, so we have to count the first two terms of the other series separately.

$$[2r(r-1)a_0+ra_0]t^{r-1}+[2(1+r)ra_1+(1+r)a_1]t^r+\sum_{n=2}^{\infty}[2(n+r)(n+r-1)a_n+(n+r)a_n+a_{n-2}]t^{n+r-1}=0$$

Setting the coefficients of each power of $t$ equal to zero gives

- $2r(r-1)a_0 + ra_0 = r(2r-1)a_0 = 0$
- $2(1+r)ra_1 + (1+r)a_1 = (r+1)(2r+1)a_1 = 0$
- $2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2} = 0$, or equivalently

$$(n+r)[2(n+r)-1]a_n = -a_{n-2}, \qquad n \geq 2$$

With the first equation we compute $r = 0$ or $r = 1/2$. In any case, the second equations forces $a_1$ to be equal to zero. We now have to determine $a_n$ for $n \geq 2$ (as function of the parameter $a_0 \neq 0$).

- Case $r = 0$. In this case the third equation becomes

$$a_n = \frac{-a_{n-2}}{n(2n-1)}, \qquad n \geq 2$$

But $a_1 = 0$, which means that using the previous recurrence, all the odd coefficients are zero. We compute the even coefficients on by one:

$$a_2 = \frac{-a_0}{2\cdot 3}, \quad a_4 = \frac{a_2}{4\cdot 7} = \frac{a_0}{2\cdot 4\cdot 3\cdot 7}, \quad a_6 = \frac{-a_4}{6\cdot 11} = \frac{-a_0}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11}$$

We keep proceeding this way and setting $a_0 = 1$, and we generalize saying that

$$y_1(t) = 1 - \frac{t^2}{2\cdot 3} + \frac{t^4}{2\cdot 4\cdot 3\cdot 7} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{2^n n! 3\cdot 7\cdots(4n-1)}$$

is one solution of our equation. It's easy to check that the interval of convergence of this series is all the real line.

- Case $r = 1/2$. In this case the recurrence is

$$a_n = \frac{-a_{n-2}}{(n+1/2)[2(n+1/2)-1)]} = \frac{-a_{n-2}}{n(2n+1)}, \qquad n \geq 2$$

Since $a_1 = 0$, all the rest of the odd coefficients is zero. We determine the even coefficients as

$$a_2 = \frac{-a_0}{2\cdot 5}, \quad a_4 = \frac{a_2}{4\cdot 9} = \frac{a_0}{2\cdot 4\cdot 5\cdot 9}, \quad a_6 = \frac{-a_4}{6\cdot 13} = \frac{-a_0}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13}$$

Setting $a_0 = 1$ we find that

$$y_2(t) = t^{1/2}\left[1 - \frac{t^2}{2\cdot 5} + \frac{t^4}{2\cdot 4\cdot 5\cdot 9} + \dots\right] = t^{1/2}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n t^{2n}}{2^n n! 5\cdot 9\cdots(4n+1)}\right]$$

is another solution of our equation and it's a trivial exercise to check that they are linearly independent. □