MGT

my geeky tutor Calculus

Problem 1. Find the sum of the series

$$\sum_{k=0}^{\infty} \frac{1}{k(k+1)}$$

if the limit exists.

Solution. Using the definition of the limit of a series, we use *partial sums* to find the number we are looking for. By definition we have:

$$\sum_{k=0}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left[\sum_{k=0}^{n} \frac{1}{k(k+1)} \right]$$

This is simply by definition of the sum of a series. Then we note that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

We replace in the previous equation to get:

$$\sum_{k=0}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

Now, we calculate

$$\sum_{k=0}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

It's easy to observe that we can cross out most of the terms in the last sum, the only "survivors" are the first and the last. So we find that:

$$\sum_{k=0}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}$$

So finally, we replace in the second equation to get

$$\sum_{k=0}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1$$